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Wess–Zumino–Witten model off criticality

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Abstract. We study the renormalization-group flow properties of the Wess–Zumino–Witten model in the region of couplings between $g^2 = 0$ and $g^2 = 4\pi/k$ by evaluating the two-loop Zamolodchikov c-function. We also discuss the region of negative couplings.

In this paper we study the Wess–Zumino–Witten (WZW) model using perturbation theory on a manifold with plane topology.

We construct Zamolodchikov's c-function from its original definition [1] up to two loops and show that it fulfils the expected properties except the stationarity condition at the trivial fixed point (FP).

We show that in the region of couplings between the trivial FP $g^2 = 0$ and the well known IR stable FP $g^2 = 4\pi/k$, our c-function monotonically decreases along a renormalization-group trajectory (for large values of k) and takes the value of the Virasoro central charges (VCC) of the critical theory at the corresponding FPs. It is stationary at the point $g^2 = 4\pi/k$, but not at the point $g^2 = 0$. In the region of negative couplings between $g^2 = -4\pi/k$ and $g^2 = 0$, where the theory is non-unitary, the c-function increases towards the IR. It takes the values of the VCC at the FPs and is stationary at the point $g^2 = -4\pi/k$, but not at the trivial point.

We also show that our expression for the c-function coincides with the expression that can be obtained from the generalized definition proposed by Cardy [2] for curved manifolds when applied to the Wess–Zumino model in S^2 [3]. This is a non-trivial check since it has not been proven, in general, that this generalization of the c-function to curved manifolds fulfils the expected properties.

Using the result for the β-function obtained in [4], we show that the c-function monotonically decreases towards the IR and we find the particular form of the coefficient which relates the β-function to the derivative of the c-function in our regularization scheme. (This coefficient cannot be set equal to one in an arbitrary regularization scheme [5].) We find that in our case, this coefficient is positive definite.

The action for the WZW model is given by [6]

$$W[h] = \frac{1}{2g^2}\int \mathrm{d}^2x\ \mathrm{tr}(\partial_\mu h^{-1}\partial_\mu h) + \frac{k}{12\pi}\int \mathrm{d}^3y\,\epsilon_{ijk}\,\mathrm{tr}(h^{-1}\partial_i h h^{-1}\partial_j h h^{-1}\partial_k h) \tag{1}$$

where h takes values in some compact Lie group G.

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Besides the trivial FP $g^2 = 0$, we know that this model has an exact non-trivial (IR stable) FP at

$$g^2 = \frac{4\pi}{k} \tag{2}$$

as was shown in [6, 7] using symmetry arguments.

Perturbative evaluations of the β-function have been performed by several authors [4–8] and the results show the existence of this FP at each order in perturbation theory. Also, the VCC of the model at this point are known to be exactly given by

$$c = \frac{k \dim G}{k + C_G} \tag{3}$$

where k is the level of the Kac–Moody algebra and C_G is the quadratic Casimir of G.

In the region of negative couplings, there is another point at which the theory becomes conformally invariant [6]; it corresponds to

$$g^2 = -\frac{4\pi}{k} \qquad k > 0 \tag{4}$$

with the VCC given by

$$c = \frac{-k \dim G}{-k + C_G}. \tag{5}$$

The corresponding theory is non-unitary and must be quantized with an indefinite metric. We shall also consider this point since, in the study of coset models (for both bosonic [9, 10] and fermionic [11] descriptions), WZW models with negative Kac–Moody central charge do appear. (In these cases, one has a Becchi–Rouet–Stora–Tyupkin quantization condition which avoids the appearance of negative norm states in the physical spectrum [10].)

We now evaluate Zamolodchikov's c-function perturbatively using its original definition and study the two regions corresponding to g^2 in $[-4\pi/k, 0]$ and g^2 in $[0, 4\pi/k]$.

The Zamolodchikov c-function is defined by [1]

$$c_{\mathrm{Zam}}(g^2) = [2z^4\langle T(x)T(0)\rangle + 4z^2x^2\langle T(x)\Theta(0)\rangle - 6x^4\langle\Theta(x)\Theta(0)\rangle]|_{x^2=R^2} \tag{6}$$

where $z = x_0 + \mathrm{i}x_1$, $\bar{z} = x_0 - \mathrm{i}x_1$, $T = T_{zz}$ and $\Theta = 4T_{z\bar{z}}$ are the two independent components of the energy–momentum tensor in these coordinates (R is a normalization point). In order to make contact with Zamolodchikov's original construction, we shall evaluate equation (6) by formulating the theory on a curved worldsheet in a generally covariant way. We then obtain the correlators appearing in (6) by differentiating the effective action with respect to the background metric $\gamma_{\alpha\beta}$ and, finally, take the limit $\gamma_{\alpha\beta} \to \delta_{\alpha\beta}$.

To this end, we define the effective action in a background metric as usual:

$$\mathrm{e}^{-S_{\mathrm{eff}}[\gamma]} = \int Dh\, \mathrm{e}^{-W[h;\gamma]} \tag{7}$$

and evaluate its finite part perturbatively up to two loops using dimensional regularization in an arbitrary metric [12]. In order to avoid IR divergences, we must include a mass term [4] but it can be shown that this does not affect the c-function.

The calculations are analogous to those performed for the Gross–Neveu model (see [13]). For the one-loop effective action we get a trivial result which corresponds to a free bosonic action with no scale dependence, thus, leading to a constant c-function.

The full expression for the finite part of the two-loop effective action is

$$S_{\mathrm{eff}}\{\gamma\} = (N^2-1)D\{\gamma\}\left[1-\frac{Ng^2}{8\pi}\left(3-\left(\frac{g^2k}{4\pi}\right)^2\right)+\cdots\right] \tag{8}$$

where

$$D\{\gamma\} = \tfrac{1}{2}[\ln\det(-\nabla^2)]. \tag{9}$$

(The finite effective action differs in no essential way from that calculated on a manifold with the topology of a sphere [3].)

In the case at hand, the determinant of the Laplacian operator is given by [14]

$$D\{\gamma\} = -\frac{1}{48\pi}\int \mathrm{d}^2x\,\mathrm{d}^2y\,\sqrt{\gamma(x)}\sqrt{\gamma(y)}R(x)R(y)G(x,y)+c\int \mathrm{d}^2x\,\sqrt{\gamma(x)} \tag{10}$$

where $G(x, y)$ is the Green function of the covariant Laplacian

$$\partial_\mu\left(\frac{1}{\sqrt{\gamma}}\gamma^{\mu\nu}\partial_\nu\right)G(x,y) = \delta(x,y). \tag{11}$$

The connected part of the general correlator of two energy–momentum operators is defined through

$$\begin{aligned}\langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle - \langle T_{\mu\nu}(x)\rangle\langle T_{\rho\sigma}(y)\rangle &= -\frac{2}{\sqrt{\gamma(x)}}\frac{2}{\sqrt{\gamma(y)}}\frac{\delta^{(2)}S_{\mathrm{eff}}[\gamma]}{\delta\gamma^{\mu\nu}(x)\delta\gamma^{\rho\sigma}(y)}\bigg|_{\gamma^{\mu\nu}=\delta^{\mu\nu}} \\ &\quad + \frac{2}{\sqrt{\gamma(x)}}\frac{2}{\sqrt{\gamma(y)}}\left\langle < \frac{\delta^{(2)}W[h;\gamma]}{\delta\gamma^{\mu\nu}(x)\delta\gamma^{\rho\sigma}(y)}\right\rangle\bigg|_{\gamma^{\mu\nu}=\delta^{\mu\nu}}\end{aligned} \tag{12}$$

where

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma^{\mu\nu}}S_{\mathrm{eff}}[\gamma]. \tag{13}$$

From equations (6)–(12), it follows that (up to contact terms)

$$\begin{aligned}\langle T(x)T(0)\rangle &= \frac{1/2}{z^4}c(g^2,k) \\ \langle T(x)\Theta(0)\rangle &= 0 \\ \langle\Theta(x)\Theta(0)\rangle &= 0\end{aligned} \tag{14}$$

where

$$c(g^2,k) = (N^2-1)\left[1-\frac{Ng^2}{8\pi}\left(3-\left(\frac{g^2k}{4\pi}\right)^2\right)+\cdots\right] \tag{15}$$

and, hence,

$$c_{\mathrm{Zam}} \equiv c(g^2, k). \tag{16}$$

It must be pointed out that there are no additional contributions to these quantities from the divergent part of the effective action [13].

Expression (16) has almost all the expected properties at the FPs given in equations (2) and (4). That is, c_{Zam} takes the correct value of the VCC at the FP

$$c(g^2, k)|_{g^2=\pm 4\pi/k} = (N^2 - 1)[1 \mp N/k + \cdots] \tag{17}$$

and it is stationary

$$\left.\frac{\partial c(g^2, k)}{\partial g^2}\right|_{g^2=\pm 4\pi/k} = 0. \tag{18}$$

It must be pointed out that the loop expansion corresponds to an expansion with $1/k$ as a small parameter. Hence, we are implicitly assuming that k is large enough and we can then study, to any desired approximation, the region of couplings between $g^2 = 0$ and $g^2 = 4\pi/k$. (We can also study separately the region of negative couplings between $g^2 = -4\pi/k$ and $g^2 = 0$.)

At the trivial FP $g^2 = 0$, we have

$$c(g^2, k)|_{g^2=0} = N^2 - 1 \tag{19}$$

which corresponds to the value of the VCC of $N^2 - 1$ free massless bosons, as expected; however, the stationarity condition is not fulfilled at this point

$$\left.\frac{\partial c(g^2, k)}{\partial g^2}\right|_{g^2=0} \neq 0. \tag{20}$$

(This subtle point deserves further investigation but will not be discussed in this paper.)

In order to study the renormalization-group behaviour of the c-function, we use the result for the β-function quoted in [4]

$$\beta(g^2) = -\frac{Ng^4}{2\pi}\left[1 - \left(\frac{g^2 k}{4\pi}\right)^2\right] + \cdots \tag{21}$$

which means that $g^2(\mu)$ is increasing towards the IR; correspondingly, the Zamolodchikov's c-function is seen to decrease along a renormalization-group trajectory for $g^2 > 0$ as expected.

If g^2 is allowed to take negative values then our expression for the c-function shows that it increases towards the IR and is stationary at the non-trivial FP $g^2 = -4\pi/k$. (This increase in the c-function does not contradict the c-theorem since for negative couplings the model is non-unitary and, hence, Zamolodchikov's proof does not hold.)

We can also calculate the coefficient which relates the β-function to the derivative of the c-function $F(g^2) = \beta(g^2)/(\partial c(g^2, k)/\partial g^2)$; in the above approximation it is given by

$$F[g^2] = \tfrac{2}{3} g^4 (N^2 - 1)^{-1} \tag{22}$$

and is positive definite as predicted in [5]. This explains why the vanishing of the β-function at $g^2 = 0$ does not necessarily lead to the stationarity of c at this FP.

As mentioned in the introduction, a generalization of the c-theorem to curved manifolds was proposed in [2]. In that paper, it is suggested that a natural definition, which could be useful in the generalization of the c-theorem to four dimensions, is given by

$$\tilde{c} = -3 \int_{S^2} \langle \Theta \rangle \sqrt{\gamma}\, \mathrm{d}^2 x \tag{23}$$

where the integration is performed over the sphere S^2 endowed with the metric induced by the embedding into R^3.

The vacuum expectation value (VEV) $\langle \Theta(x) \rangle$ has been evaluated exactly in [3] at the FP (2). In order to evaluate it off criticality, one has to look not only at the finite part of the effective action, but also at its divergent part (which behaves as $1/(d-2)$ where d is the dimensionality of spacetime) since it can give a non-trivial finite contribution to the trace of the energy–momentum tensor. However, in the present case it is easy to show that the trace receives no additional contributions arising from the divergent terms in the effective action; the result being simply

$$\langle \Theta \rangle \equiv \gamma^{\mu\nu}(x) \langle T_{\mu\nu}(x) \rangle = c(g^2, k) \frac{R(x)}{24\pi} \tag{24}$$

where $R(x)$ is the scalar curvature and $c(g^2, k)$ is given by equation (15).

Inserting equation (24) into equation (23) gives

$$\tilde{c} \equiv c(g^2, k) \tag{25}$$

and we recover equation (16). Hence, we conclude that, in the present case, both definitions coincide. This equality must be taken with 'a pinch of salt' since there does not exist a complete proof demonstrating that this generalized 'c-function' fulfils the same conditions as Zamolodchikov's c-function. In particular, the decreasing property of this function has not been proven, although it was verified to lowest order in perturbation theory [2].

As a final comment, it is interesting to note that the effective action, (8) and (9), can be written as

$$\mathrm{e}^{-S_{\mathrm{eff}}[\gamma]} = [\det(-\nabla)]^{-\frac{1}{2} c(g^2, k)} \tag{26}$$

which is the $c(g^2, k)$-power of the partition function for a massless free boson. This fact agrees with the intuition that the c-function is a measure of the massless degrees of freedom [1].

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